Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Generalized Nash Equilibrium Problems with Applications to Spot Markets with Gas Transport

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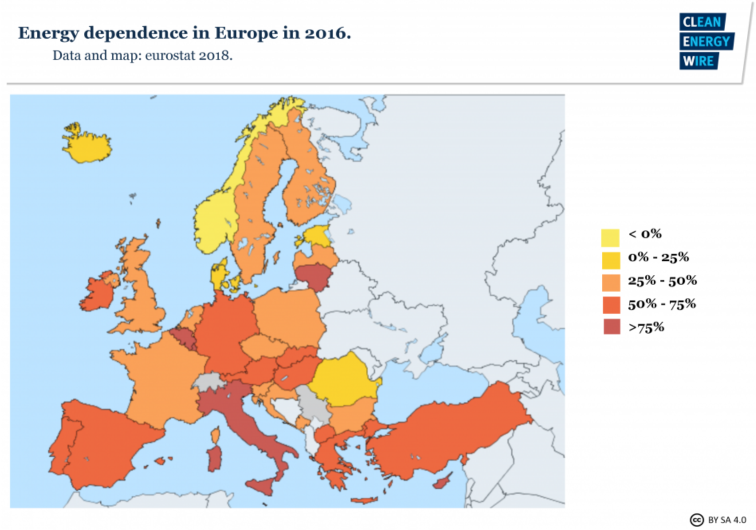
joint work with V. Grimm, O. Huber, T. M. Surowiec, A. Kämmler, L. Schewe, M. Schmidt, G. Zöttl


Energy dependence in Europe in 2016.
Data and map: eurostat 2018.
CLEAN ENERGY WIRE
< 0%
0% - 25%
25% - 50%
50% - 75%
>75%
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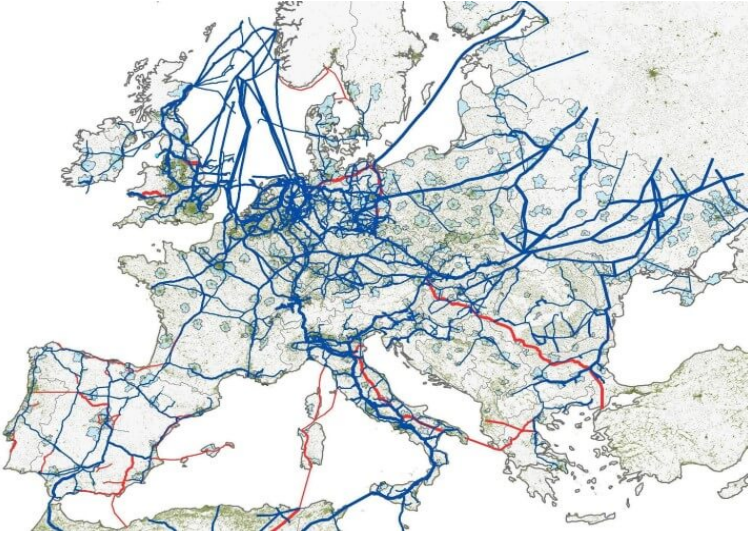
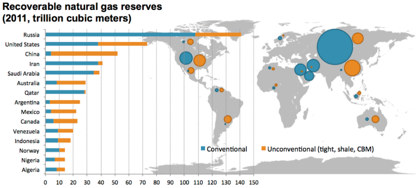


Map created by ETH Zurich, 2014

Why focus on (natural) gas?

Energy turnaround: away from nuclear (GER: early 2020ies) and fossil fuels (GER: around 2050) to renewables.

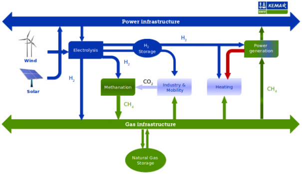
wordpress.com

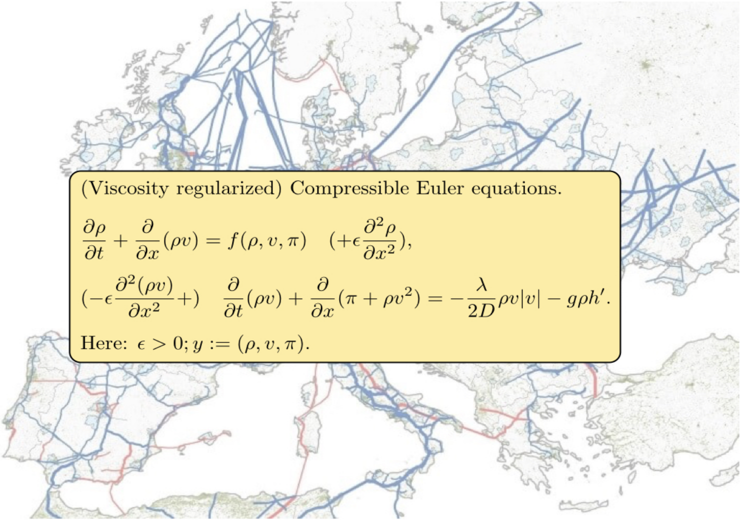


Sources: IEA (Advanced Resources International), BP, Reuters

Availability (still) of natural gas.

Transport, storage, distribution and conversion (Power to Gas!).



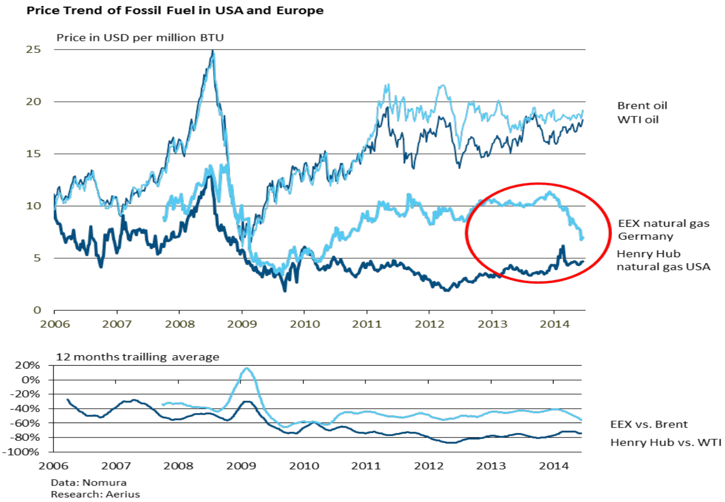


(Viscosity regularized) Compressible Euler equations.

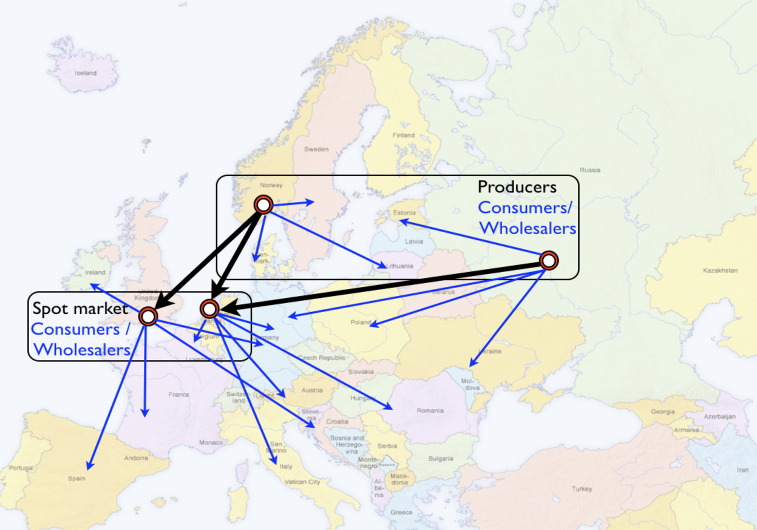
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = f(\rho, v, \pi) \quad (+\epsilon \frac{\partial^2 \rho}{\partial x^2}),$$

$$(-\epsilon \frac{\partial^2 (\rho v)}{\partial x^2} +) \quad \frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\pi + \rho v^2) = -\frac{\lambda}{2D}\rho v|v| - g\rho h'.$$

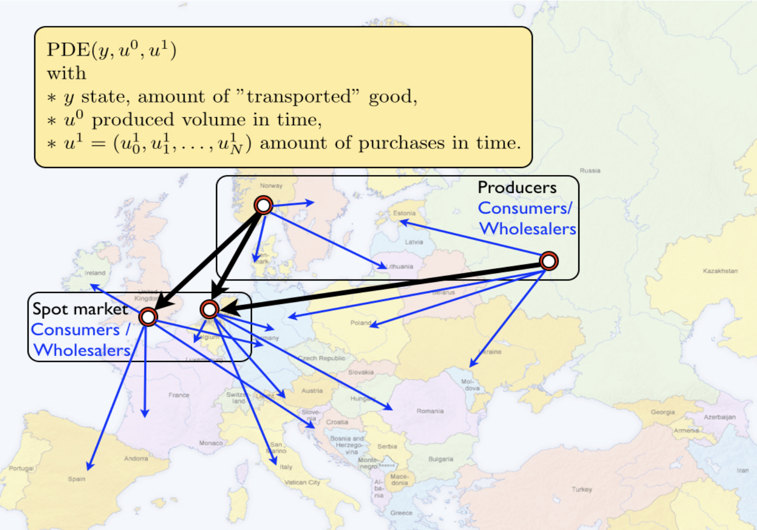
Here: $\epsilon > 0; y := (\rho, v, \pi)$.


Price Trend of Fossil Fuel in USA and Europe
Price in USD per million BTU
25
20
15
10
5
0
2006
2007
2008
2009
2010
2011
2012
2013
2014
Brent oil
WTI oil
EEX natural gas Germany
Henry Hub natural gas USA
12 months trailling average
20%
0%
-20%
-40%
-60%
-80%
-100%
EEX vs. Brent
Henry Hub vs. WTI
Data: Nomura
Research: Aerius

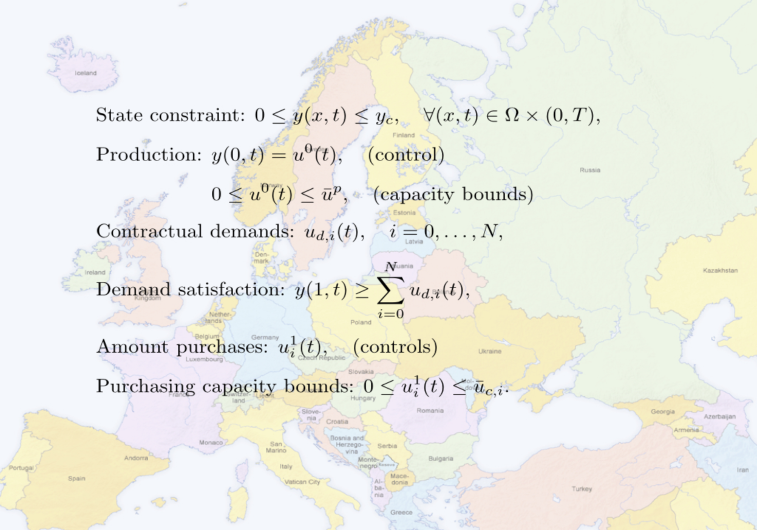
Spot market with transport (of energy carriers)



- Producers and wholesalers are exchanging goods via a pipe/road in a non-cooperative fashion
- Evolution w.r.t time and space of the goods is governed by a PDE, which is a shared constraint
- Oligopoly case: wholesalers are price takers and they make decision at each point in time
- The remaining players in the game are the producers
- The coupling between players happens via the PDE and the objective functions



PDE(y, u^0, u^1)
with
$*$ y state, amount of "transported" good,
$*$ u^0 produced volume in time,
$*$ $u^1 = (u^1_0, u^1_1, \dots, u^1_N)$ amount of purchases in time.
Producers
Consumers/
Wholesalers
Spot market
Consumers /
Wholesalers



State constraint: $0 \leq y(x,t) \leq y_c, \quad \forall (x,t) \in \Omega \times (0,T),$

Production: $y(0,t) = u^0(t), \quad$ (control)

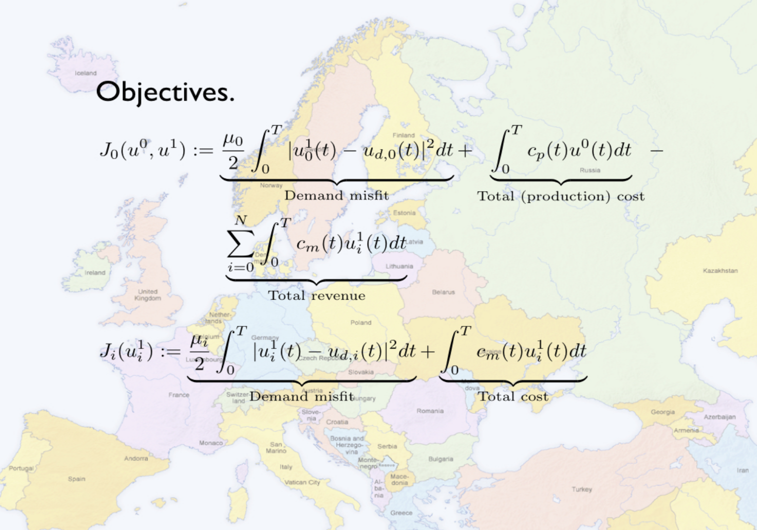
$$0 \leq u^0(t) \leq \bar{u}^p, \quad \text{(capacity bounds)}$$

Contractual demands: $u_{d,i}(t), \quad i = 0, \ldots, N,$

Demand satisfaction: $y(1,t) \geq \sum_{i=0}^{N} u_{d,i}(t),$

Amount purchases: $u_i^1(t), \quad$ (controls)

Purchasing capacity bounds: $0 \leq u_i^1(t) \leq \bar{u}_{c,i}.$



Objectives.

$$J_0(u^0,u^1) := \underbrace{\frac{\mu_0}{2}\int_0^T |u_0^1(t) - u_{d,0}(t)|^2 dt}_{\text{Demand misfit}} + \underbrace{\int_0^T c_p(t)u^0(t)dt}_{\text{Total (production) cost}} - \underbrace{\sum_{i=0}^{N}\int_0^T c_m(t)u_i^1(t)dt}_{\text{Total revenue}}$$

$$J_i(u_i^1) := \underbrace{\frac{\mu_i}{2}\int_0^T |u_i^1(t) - u_{d,i}(t)|^2 dt}_{\text{Demand misfit}} + \underbrace{\int_0^T c_m(t)u_i^1(t)dt}_{\text{Total cost}}$$

General Nash Equilibrium Problem - GNEP

Producer's problem.

$$\begin{aligned} &\text{minimize } J_0(u^0, u^1) \text{ over } (u^0, u_0^1) \\ &\text{subject to } PDE(y, u^0, u^1) \ + \text{state constraints} \\ &\qquad 0 \le u_0^1 \le \bar{u}_{c,0}, \quad 0 \le u^0 \le \bar{u}_p. \end{aligned}$$

i-th consumer's problem (i=1,...,N).

$$\begin{aligned} &\text{minimize } J_i(u_i^1) \text{ over } u_i^1 \\ &\text{subject to } PDE(y, u^0, u^1) + \text{state constraints} \\ &\qquad 0 \le u_i^1 \le \bar{u}_{c,i}. \end{aligned}$$

∗Possible multiplicity of solutions coming from the shared constraints motivates the restriction to Variational Equilibrium (VE).

∗Meaningful economical interpretations in the VE case.

- Abstract GNEPs in Banach space.
- Existence of solutions and equilibrium conditions.
- Nikaido-Isoda based path-following.
- Numerical results.
- Outlook on spot market model.

Aim of player $i = 1, \ldots, N$: Given u_{-i}, choose (u_i, y) which solves:

$$
\begin{aligned}
&\min J_i^1(y) + J_i^2(u_i) \text{ over } (u_i, y) \in U_i \times Y \\
&\text{subject to (s.t.)} \\
&\qquad Ay = B(u_i, u_{-i}), \\
&\qquad u_i \in U_{\mathrm{ad}}^i, \\
&\qquad y \in K.
\end{aligned}
\tag{P$_i$}
$$

Data assumptions

- U_i $(i = 1, \ldots, N)$ reflexive separable Banach spaces, $U := \Pi_{i=1}^N U_i$.
- $A : Y \to W$ linear isomorphism; with Y, W reflexive B.-spaces.
- X B.-space with $Y \hookrightarrow X$ is continuous.
- If $M \subset X^*$ is bounded, then M weak-$*$ relatively compact in X^*.
- $B \in \mathcal{L}(U, W)$; $Bu = \sum_{i=1}^m B_i u_i$ with $B_i = B(\cdot, 0_{-i})$ with $B_i \in \mathcal{L}(U_i, W)$.
- $A^{-1}B : U \to X$ is compact.

- $K \subset X$ nonempty, closed, and convex set.
- Norm topology on X: $\exists\, y_0 \in K$ and $\varepsilon > 0$: $\mathbb{B}_\varepsilon(y_0) \subset K$.

- $U^i_{\text{ad}} \subset U_i$ nonempty, bounded, closed, and convex; and $U_{\text{ad}} := \Pi_{i=1}^N U^i_{\text{ad}}$.
- $\exists\, u \in U_{\text{ad}}$ with $A^{-1}Bu \in K$.
- $J^1_i : Y \to \mathbb{R}$ convex and completely continuous (if $v_k \overset{Y}{\rightharpoonup} v$, then $J^1_i(v_k) \to J^1_i(v)$), and $J^2_i : U_i \to \mathbb{R}$ strictly convex and continuous.

Reduced form using solution operator $\mathbf{S} : \mathbf{U} \to \mathbf{Y}$, $Su := A^{-1}(Bu)$:

$$\min \mathcal{J}_i(u_i, u_{-i}) := J_i^1(S(u_i, u_{-i})) + J_i^2(u_i) \text{ over } u_i \in U_i$$

s.t.

$$u_i \in U_{\mathrm{ad}}^i, \quad S(u_i, u_{-i}) \in K.$$

For $u \in U$ strategy u_i feasible for ith problem,

given u_{-i}, for all $i = 1, \ldots, N$ if and only if $u \in C$, where

$$C := \{u \in U_{\mathrm{ad}} \mid Su \in K\}.$$

Since C convex, problem structure of so-called jointly convex GNEP.

Definition (Generalized Nash Equilibrium)

$\bar{u} \in C$ is Nash equilibrium provided

$$\mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) \leq \mathcal{J}_i(v_i, \bar{u}_{-i}), \ \forall v_i \in U_i : (v_i, \bar{u}_{-i}) \in C, \ \forall i = 1, \ldots, N.$$

Major complications:

- **Existence:** Classical (Ky Fan/Kakutani) theorems not directly applicable.
  ($\Rightarrow$ resort to weak topology).
- **Equilibria:** Generalized Nash vs. more tractable variational equilibria.
  ($\Rightarrow$ consider variational equilibria).

- **Numerical approach**: Handling of state constraints.
  ($\Rightarrow$ Moreau-Yosida regularization).
- **Update of path parameter**: Primal-dual path-following strategy.
  ($\Rightarrow$ Nikaido-Isoda function).

- **Finite dimensions.** Much work done for generalized Nash equilibrium problems (GNEPs); see works by Facchinei, Kanzow, Pang, Fukushima, and many more.

- **Infinite dimensions.** Significantly less in function spaces: Desideri; Hoppe; Ramos, Glowinski, & Periaux; Ramos & Roubicek; Kanzow, Karl, Steck, D. Wachsmuth; Borzì,...

  Often: multi-objective – monotone VI, but not GNEP (!); in some cases NEP.

Optimality Conditions for Generalized Nash Equilibria

- If a Nash equilibrium $\bar{u} \in U$ of (P) satisfies

$$\forall i = 1, \ldots, N, \exists u_i \in U_{\text{ad}}^i : \quad \mathbb{B}_\varepsilon(0) \subset S(u_i, \bar{u}_{-i}) - K$$

  for some $\varepsilon > 0$, then $\exists\, \bar{y} \in Y, \bar{p} \in (W^*)^N, \bar{\lambda} \in U^*$ and $\bar{\mu} \in (X^*)^N$:

$$(\text{OS}_i) \begin{cases} \bar{y} &= S\bar{u}, \\ -\bar{p}_i &\in A^{-*}\big(\partial J_i^1(\bar{y}) + \bar{\mu}_i\big), \\ \bar{\lambda}_i &\in \partial I_{U_{\text{ad}}^i}(\bar{u}_i), \\ \bar{\mu}_i &\in \partial I_K(\bar{y}), \\ 0 &\in \partial J_i^2(\bar{u}_i) - B_i^* \bar{p}_i + \bar{\lambda}_i, \end{cases}$$

  is fulfilled for $i = 1, \ldots, N$. Coupled system is denoted by (OS).
- Conversely, if the tuple $(\bar{u}, \bar{y}, \bar{p}, \bar{\lambda}, \bar{\mu}) \in U \times Y \times (W^*)^N \times U^* \times (X^*)^N$ satisfies the coupled system (OS), then $\bar{u}$ is a Nash equilibrium.

Nikaido-Isoda Function

- Nikaido-Isoda function $\Psi : U \times U \to \mathbb{R}$ defined by

$$\Psi(u,v) := \sum_{i=1}^{N} \left[\mathcal{J}_i(u_i, u_{-i}) - \mathcal{J}_i(v_i, u_{-i})\right].$$

- In addition, define $V : C \to \mathbb{R}$ by

$$V(u) = \max_v \{\Psi(u,v) \mid v \in U : (v_i, u_{-i}) \in C \text{ for } i = 1, \dots, N\}.$$

- Observation: For $v = u$ we get $V(u) \geq \Psi(u,u) = 0$ for $u \in C$.

A point $\bar{u} \in U$ is a Nash equilibrium of (P) if and only if $\bar{u} \in C$ and $V(\bar{u}) = 0$.

- Since (P) is a jointly-convex **GNEP** we can use the more restrictive solution concept of variational equilibria ([Rosen '65]).
- For **NEPs**, variational and Nash equilibria coincide (i.e., $K = Y$).
- Define $\widehat{\mathcal{R}} : C \to C$ by

$$\widehat{\mathcal{R}}(u) := \operatorname*{argmax}_v \{\Psi(u,v) \mid v \in C\} = \operatorname*{argmin}_v \left\{\sum_{i=1}^N \mathcal{J}_i(v_i, u_{-i}) \mid v \in C\right\}$$

and $\widehat{V} : C \to \mathbb{R}$ by

$$\widehat{V}(u) := \Psi(u, \widehat{\mathcal{R}}(u)) = \max_v \{\Psi(u,v) \mid v \in C\}.$$

A point $\bar{u} \in U$ is called a variational equilibrium of (P) if $\bar{u} \in C$ and $\widehat{V}(\bar{u}) = 0$.

Variational Equilibria are Nash Equilibria

Every variational equilibrium of (P) is also a Nash equilibrium of (P).

A point $\bar{u} \in C$ is a variational equilibrium if and only if $\bar{u} = \widehat{\mathcal{R}}(\bar{u})$.

Existence

The GNEP (P) admits a variational equilibrium $\bar{u} \in U$.

Proof uses Kakutani's Fixed Point Theorem applied to weak topology (yields compactness of C and upper semicontinuity of set-valued map $\widehat{\mathcal{R}}$, which then has a fixed point).

$\Rightarrow$ (P) admits Nash equilibrium.

Slater constraint qualification (weaker than previous CQ.)

$$0 \in \text{int}\left(S(U_{\text{ad}}) - K\right), \quad \text{interior taken in } X.$$

First-order optimality conditions

Suppose Slater CQ satisfied. Then $\bar{u} \in U$ is variational equilibrium of (P) if and only if $\exists$ $\bar{y} \in Y$, $\bar{p} \in (W^*)^N$, $\bar{\lambda} \in U^*$ and $\bar{\mu} \in X^*$ such that

$$(\widehat{\text{OS}_i}) \begin{cases} \bar{y} &= S\bar{u}, \\ -\bar{p}_i &\in A^{-*}\left(\partial J_i^1(\bar{y}) + \bar{\mu}\right), \\ \bar{\lambda}_i &\in \partial I_{U_{\text{ad}}^i}(\bar{u}_i), \\ \bar{\mu} &\in \partial I_K(\bar{y}), \\ 0 &\in \partial J_i^2(\bar{u}_i) - B_i^*\bar{p}_i + \bar{\lambda}_i, \end{cases}$$

is fulfilled for each $i = 1, \ldots, N$. Coupled system referred to by $(\widehat{\text{OS}})$.

Structural assumption.

- $J_i^1 = J_0^1 + \tilde{J}_i^1$ where J_0^1 convex and continuously Gâteaux differentiable, and $\tilde{J}_i^1$ linear-affine; w. l. o. g. we assume $\tilde{J}_i^1 \in Y^*$.

Includes typical tracking-type functionals: $J_i^1(y) = \frac{1}{2}\|y - y_i^d\|_Y^2$. Since,

$$\frac{1}{2}\|y - y_i^d\|_{L^2}^2 = \frac{1}{2}\|y\|_{L^2}^2 - (y, y_d^i)_{L^2} + \frac{1}{2}\|y_d^i\|_{L^2}^2.$$

Single objective PDE constrained optimization (— potential game)

Under the above assumption there exists a unique variational equilibrium $\bar{u}$ of (P), which is the unique solution of the convex optimization problem

$$\text{minimize } \widehat{J}(u) := J_0^1(Su) + \sum_{i=1}^{N} \left(J_i^2(u_i) + \langle S_i^* \tilde{J}_i^1, u_i \rangle_{U_i^*, U_i} \right) \text{ over } u \in U.$$

s.t. $u \in C$.

Penalty function (e.g. L^2-type Moreau-Yosida regularization of indicator of K).

- $\beta : X \to \mathbb{R}_+$ is convex, continuous, and cont. Gâteaux-differentiable with $\ker \beta = K$, i.e., $\beta(y) = 0$ whenever $y \in K$, else $\beta(y) > 0$.

Consider

$$\begin{aligned} &\min J_i^1(y) + J_i^2(u_i) + \gamma\beta(y) \text{ over } (u_i, y) \in U_i \times Y \\ &\text{s.t.} \\ &\qquad Ay = B(u_i, u_{-i}), \quad u_i \in U_{\mathrm{ad}}^i. \end{aligned} \tag{P$_{i,\gamma}$}$$

First-order conditions.

For all $i = 1, \ldots, N$, u^γ is a Nash equilibrium if and only if there exist $y^\gamma \in Y$, $p^\gamma \in (W^*)^N$, $\lambda^\gamma \in U^*$ and $\mu^\gamma \in X^*$ such that

$$(\mathrm{OS}_{i,\gamma}) \begin{cases} y^\gamma = Su^\gamma, \\ -p_i^\gamma = A^{-*}\left((J_i^1)'(y^\gamma) + \mu^\gamma\right), \\ \lambda_i^\gamma \in \partial I_{U_{\mathrm{ad}}^i}(u_i^\gamma), \\ \mu^\gamma = \gamma\beta'(y^\gamma), \\ 0 = (J_i^2)'(u_i^\gamma) - B_i^* p_i^\gamma + \lambda_i^\gamma. \end{cases}$$

For $\gamma > 0$, $\mathcal{S}_\gamma \subseteq U \times Y \times (W^*)^N \times U^* \times X^*$ set of solutions of (OS_γ).

$$\mathbf{C} := \left\{ \left((u^\gamma, y^\gamma, p^\gamma, \lambda^\gamma, \mu^\gamma)\right)_{\gamma>0} \mid \forall \gamma > 0 : (u^\gamma, y^\gamma, p^\gamma, \lambda^\gamma, \mu^\gamma) \in \mathcal{S}_\gamma \right\}.$$

We call every element $\mathcal{C} = \left((u^\gamma, y^\gamma, p^\gamma, \lambda^\gamma, \mu^\gamma)\right)_{\gamma>0} \in \mathbf{C}$ a primal-dual path.

Uniform Boundedness

Let (P) fulfill Slater CQ. Then $\exists\, 0 < \rho < \infty$ such that for all $\gamma > 0$:

$$\|u^\gamma\|_U + \|y^\gamma\|_Y + \|p^\gamma\|_{(W^*)^N} + \|\lambda^\gamma\|_{U^*} + \|\mu^\gamma\|_{X^*} \leq \rho.$$

Path convergence

Let (P) fulfill Slater CQ. Then for every primal-dual path $\mathcal{C} \in \mathbf{C}\ \exists\, \gamma_n \to \infty$:

$$u^{\gamma_n} \overset{U}{\rightharpoonup} u^*,\ y^{\gamma_n} \overset{Y}{\rightharpoonup} y^*,\ p^{\gamma_n} \overset{(W^*)^N}{\rightharpoonup} p^*,\ \lambda^{\gamma_n} \overset{U^*}{\rightharpoonup} \lambda^*,\ \mu^{\gamma_n} \overset{X^*}{\rightharpoonup}{}^* \mu^*.$$

Moreover, the point $(u^*, y^*, p^*, \lambda^*, \mu^*)$ fulfills the first order optimality conditions $(\widehat{\mathrm{OS}})$, in particular u^* is a variational equilibrium of (P).

- For any $\gamma > 0$ let $\Psi_\gamma : U \times U \to \mathbb{R}$ be the Nikaido-Isoda function for (P$_\gamma$), i.e.

$$\Psi_\gamma(u,v) := \sum_{i=1}^{N} \left[\mathcal{J}_i^\gamma(u_i,u_{-i}) - \mathcal{J}_i^\gamma(v_i,u_{-i})\right],$$

  where $\mathcal{J}_i^\gamma(u) := J_i^1(Su) + J_i^2(u) + \gamma\beta(Su)$ represents the objective of (P$_{i,\gamma}$), and

- consider $V : U \times \mathbb{R}_+ \to \mathbb{R}$ defined by

$$V(u,\gamma) := \max_{v \in U_{\text{ad}}} \Psi_\gamma(u,v) = \sum_{i=1}^{N} \mathcal{J}_i^\gamma(u_i,u_{-i}) - \min_{v \in U_{\text{ad}}} \sum_{i=1}^{N} \mathcal{J}_i^\gamma(v_i,u_{-i}).$$

- One observes that $V(u,\gamma) \geq 0$ for all $u \in U_{\text{ad}}$ and analogously to before:

$$V(u,\gamma) = 0 \text{ if and only if } u \text{ is an equilibrium.}$$

Given NE u^γ for γ, find $\gamma_+ > \gamma$ based on deviations of $V(u^\gamma, \gamma')$ from zero.

- Let $\mathcal{V}(\gamma + \eta) := V(u^\gamma, \gamma + \eta)$, $\eta > 0$, and assume the directional derivative $\mathcal{V}'(\gamma, \eta)$ exists.
- We observe $\mathcal{V}(\gamma + t\eta) = \mathcal{V}(\gamma) + \mathcal{V}'(\gamma; t\eta) + o(t) = \mathcal{V}'(\gamma; t\eta) + o(t)$.
- Therefore, we can base either directly on $\mathcal{V}'(\gamma; \eta)$ or an efficient approximation thereof.

Estimate of dir. deriv.

For any $\gamma > 0$, let u^γ be the corresponding equilibrium and define $\mathcal{V}(\gamma + \eta) := V(u^\gamma, \gamma + \eta)$, $\eta > 0$. It holds that for all $\eta > 0$:

$$\eta N \beta(S(u^\gamma)) \geq \limsup_{t \downarrow 0} t^{-1}(\mathcal{V}(\gamma + t\eta) - \mathcal{V}(\gamma)) \geq \liminf_{t \downarrow 0} t^{-1}(\mathcal{V}(\gamma + t\eta) - \mathcal{V}(\gamma)) \geq 0.$$

- **Redundancy**: If $\beta(S(u^\gamma)) = 0$, then there is no need to increase γ, as the current state y^γ is feasible.
- **State constraint not redundant**: Bound secants by a fixed threshold $\pi_{path} > 0$ and choosing $\eta > 0$ such that

  $$\eta N \beta(S(u^\gamma)) \leq \pi_{path}.$$

  For example:

  $$\eta = \frac{\pi_{path}}{N\beta(S(u^\gamma))}$$

  and then use the update $\gamma := \gamma + \eta$.

Solvers.

- **Reducible case**. Semi-smooth Newton method (mesh independent).
- **General case**. Projected gradient-type method (subproblem SSN – mesh independent)

Viscosity regularized transport model (velocity $v \in \mathbb{R}$, $\epsilon > 0$).

$$
\begin{aligned}
y_t(x,t) + v y_x(x,t) - \epsilon y_{xx}(x,t) &= 0, \quad \text{a.e. } Q := (0,1) \times (0,T), \\
y(0,t) &= u^0(t), \quad \text{a.e. } t \in (0,T), \\
y(1,t) &= \sum_{i=0}^{N} u_i^1(t), \quad \text{a.e. } t \in (0,T), \\
y(x,0) &= y_0(x), \quad \text{a.e. } x \in (0,1).
\end{aligned}
$$

- Otherwise (constraints, objectives) the GNEP is as described in the introductory part of this presentation.

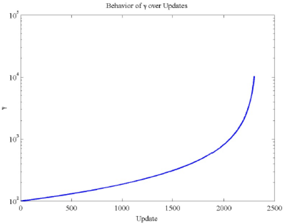
Parameter settings.

h	τ	tol	π_{path}	γ_0	ε	N	$\bar{y}_c$	$\bar{u}_c$	$\bar{u}_p$
1/256	1/200	1e-06	1e-05	1e+02	1e+00	3	3	1	3

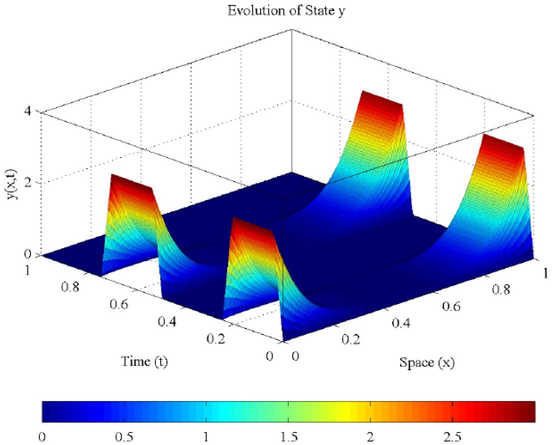
Player	0	1	2
μ	65.2883	88.4484	25.7334

Randomized, asymmetric misfit costs μ_j; periodic demands.

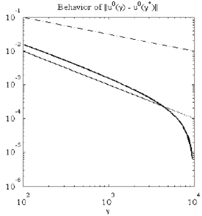
γ-updates.



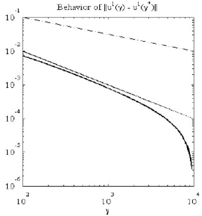
State in space-time.



Behavior of control as a function of γ.



(a)



(b)

Convergence rates of u^0 and u^1 (bold) versus $\gamma^{-1/2}$ (dashed-dotted) and γ^{-1} (dotted)

Observations.

- Wholesaler/Producer attempts to behave strategically by producing more of the product than is needed in periods of low-demand.
- Proj. grad. its.: For $\gamma \in (100, 150)$ an average of 4 iterations needed, for $\gamma \in (150, 625)$ an average of 15, for $\gamma \in (625, 2300)$ between 29 and 45 inner iterations, for $\gamma \in (2300, 5000)$ roughly 54, etc.

Oligopoly GNEP: wholesalers/consumers are price-takers.

Suppose market always clears (i.e., $\sum_i s_i = \sum_j d_j$), s_i supply and d_j demand, and a wholesalers' behavior is as follows: given a price $\pi(t)$, demand of consumer j solves

$$\max_{d_j(t)} \quad v_j(t, d_j(t)) - \pi(t) d_j(t),$$

where v_j is a strongly concave function. Hence,

$$\frac{\partial v_j(t, d_j(t))}{\partial d_j(t)} = \pi(t) \quad \text{for all} \quad t \in [0, T].$$

- Classical approach in economy relies on implicit function theorem to derive total demand $d := \sum d_j$ as a function of price π.
- This relation is then inverted to obtain the **inverse demand function** $P\colon (t, d) \mapsto \pi$, which is supposed to satisfy:

$$\frac{\partial P(t,d)}{\partial d} < 0 \quad \text{and} \quad d(t) \frac{\partial P(t,d)}{\partial d} + \frac{\partial^2 P(t,d)}{\partial d^2} \leq 0 \quad \text{a.e } (0, T). \qquad (1)$$

Let $U := L^2([0,T])$. Each producer's behavior is modeled by:

$$\max_{(u_i^p, u_i^w) \in U^2} \int_0^T \left(P\left(t, u^w(t)\right) u_i^w(t) - C_i\left(u_i^p(t)\right)\right) \mathrm{d}t$$

$$\begin{aligned} \text{such that} \quad & 0 \le u_i^p(t) \le \bar{u}_i^p, u^p(t) \le \bar{y}, u^w(t) \le \bar{y} \quad \text{a.e. on } (0,T), \\ & y := S(u^p, u^w) \text{ solution to (linear parabolic) PDE}, \\ & 0 \le y(u^p, u^w) \le \bar{y} \qquad \text{a.e. on } (0,T) \times (0,1), \\ & \int_0^T \left(u_i^w(t) - u_i^p(t)\right) \mathrm{d}t \le 0. \end{aligned}$$

Here $P(t,u) := -a(t)u(t) + b(t)$, $\bar{a} \ge a(t) > 0$ and $b \in L^\infty([0,T])$, with $u^p := \sum u_i^p$, and $u^w := \sum u_i^w$.

- **Difficulty:** Coupling terms $u_i^w \sum_j u_j^w$ when studying the best response map (i.e., optimal value function) of each player.

All N producers share the same cost functional and upper bound $\bar{u}_i^p = N^{-1}\bar{u}^p$. In this case, a solution to the variational equilibrium can be obtained from

$$\max_{(u_P^p, u_P^w, y) \in U^2 \times Y} \int_0^T P(u_P^w) u_P^w \mathrm{d}t - \frac{\kappa}{2N} \|u_P^p\|_U^2$$

$$\text{s.t.} \quad 0 \leq u_P^p(t) \leq \bar{u}^p, u_P^p(t) \leq \bar{y}, u_P^w(t) \leq \bar{y} \quad \text{a.e on } (0,T),$$

$$(y, u_P^p, u_P^w) \text{ solution to PDE},$$

$$0 \leq y(t,x) \leq \bar{y} \qquad \text{a.e on } (0,T) \times (0,1),$$

$$\int_0^T u_P^w(t)\mathrm{d}t \leq \int_0^T u_P^p(t)\mathrm{d}t.$$

Suppose that u_P is a solution, then a solution to the variational equilibrium is recovered by setting

$$u_i^p = N^{-1} u_P^p \quad \text{and} \quad u_i^w = N^{-1} u_P^w$$

for each producer.

Suppose that K is a closed bounded subset of a Banach space X and $\phi\colon X \times X \to \mathbb{R}$ satisfying

- $\forall \eta \in K, \theta \mapsto \phi(\theta, \eta)$ is weakly lower semicontinuous;
- $\forall \theta \in K, \eta \mapsto \phi(\theta, \eta)$ is concave;
- $\forall \eta \in K, \phi(\eta, \eta) \leq 0$.

Then $\exists \bar{\theta} \in K$ such that $\quad \phi(\bar{\theta}, \eta) \leq 0 \quad \forall \eta \in K$.

One can use this result to show existence of a solution to

- the VI associated with the variational equilibrium [Théra 1991];
- a point $\bar{u}$ such that $\Psi(\bar{u}) = 0$, by applying Ky Fan's inequality to the Nikaido-Isoda functional Ψ.

Summary

- Motivating application: Spot markets.
- Abstract GNEPs
  - Uniform Slater CQ, Slater CQ;
  - Ky Fan with weak topology;
  - Moreau-Yosida regularization of state constraint;
  - SNEP — Sequential NEP approach.
- Nikaido-Isoda-based primal-dual path following.
- Outlook on enriched spot market model with gas transport.

DFG SPP 1962: Priority Programme
Non-smooth and Complementarity-based Distributed Parameter Systems: Simulation and Hierarchical Optimization

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