Representability of optimization models

Amitabh Basu

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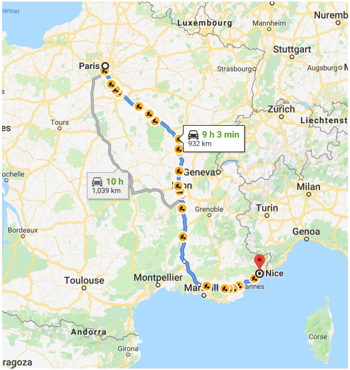
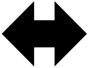
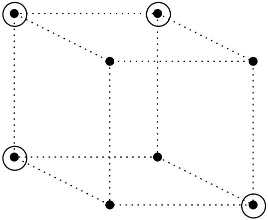
Nice, France, September 2019

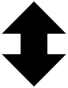
The modeling question

The optimizer's approach to making decisions:









$$Ax \leq b$$
$$x \in \{0,1\}^n$$

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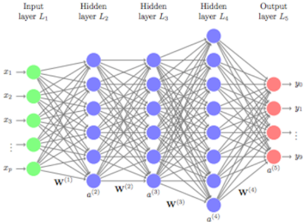

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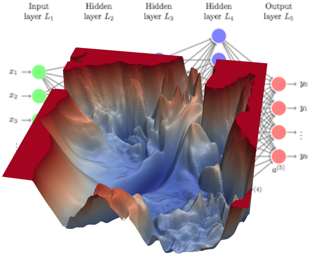

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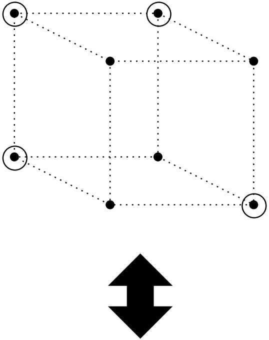
What exactly do specific optimization paradigms model?



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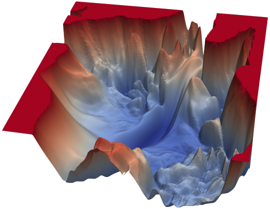

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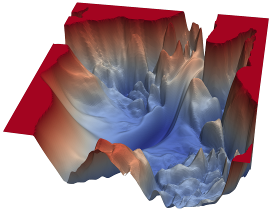

$$\Updownarrow$$

$$\left\{ x \in \mathbb{R}^n : \begin{array}{c} f_1(x, z) \leq 0 \\ \vdots \\ f_k(x, z) \leq 0 \end{array} \right\}$$

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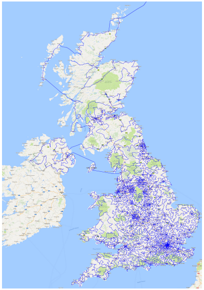
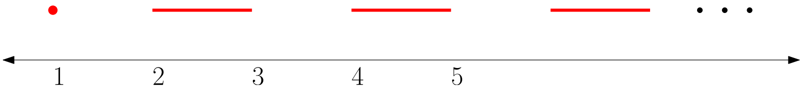


$$\left\{ x \in \mathbb{R}^{|E|} : \begin{array}{c} Ax + Bz \leq b \\ x, z \text{ mixed-integer} \end{array} \right\}$$

EXAMPLE: Miller, Tucker, Zemlin TSP formulation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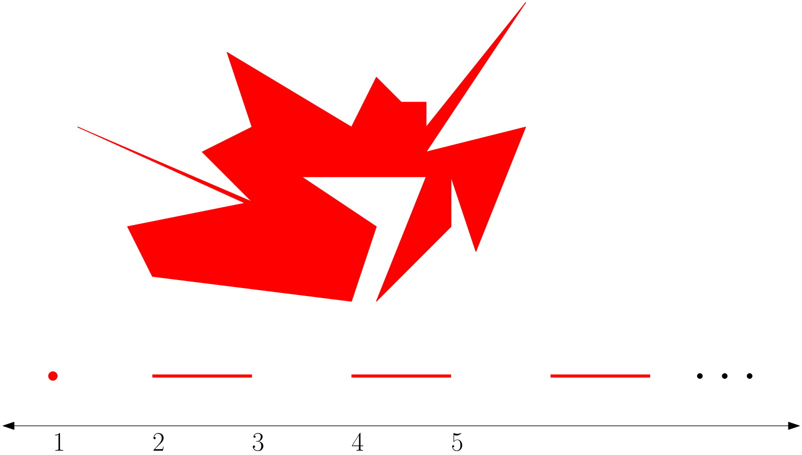
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Representability of an optimization paradigm

Given a family $\mathcal{S}$ of sets defined by an optimization family (e.g., mixed-integer linear programs), we say that a set $X \subseteq \mathbb{R}^n$ is representable by $\mathcal{S}$ if there exists $S \in \mathcal{S}$ and a linear transformation T such that $X = T(S)$.

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$$X = T(S) = \{x \in \mathbb{R}^n : x = T(s),\ \ s \in S\}$$

As long as the family $\mathcal{S}$ is closed under addition of affine constraints, projections are the same as linear transforms.

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We say that X is rationally representable by $\mathcal{S}$ if $S \in \mathcal{S}$ is described by rational data and T can be represented by a rational matrix.

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THEOREM (Fourier-Motzkin+Minkowski-Weyl): $X \subseteq \mathbb{R}^n$ is representable by $\mathcal{S}$ if and only if $X = \mathrm{conv}(V) + \mathrm{cone}(R)$ for finite sets $V, R \subseteq \mathbb{R}^n$.

Representability of Mixed-Integer Linear Programs

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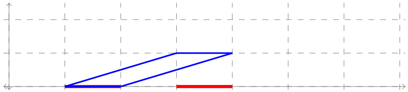

No, unless we allow projections.

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$$X = \left\{ x \in \mathbb{R} : \begin{array}{l} x = \sqrt{2} z_1 - z_2, \\ z_1, z_2 \in \mathbb{Z}_+ \end{array} \right\}$$

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THEOREM (Jeroslow-Lowe MPS 1984): $X \subseteq \mathbb{R}^n$ is rationally representable by $\mathcal{S}$ if and only if

$$X = \left(\bigcup_{i=1}^{k} P_i \right) + \mathrm{int.cone}\{r^1, \ldots, r^t\},$$

where $P_1, \ldots, P_k$ are rational polytopes and $r^1, \ldots, r^t$ are integral vectors.

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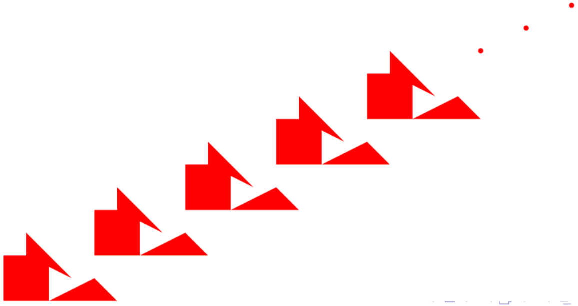

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LEMMA: Let C be a closed convex set with a rational, polyhedral recession cone generated by integral vectors $r^1, \ldots, r^t$, expressed as $C = K + \mathrm{cone}\{r^1, \ldots, r^t\}$ for some compact, convex set K. Then

$$C \cap (\mathbb{Z}^n \times \mathbb{R}^d) = \Big((K + \Pi) \cap (\mathbb{Z}^n \times \mathbb{R}^d)\Big) + \mathrm{int.cone}\{r^1, \ldots, r^t\},$$

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Proof of LEMMA: Any $(x, y) \in C \cap (\mathbb{Z}^n \times \mathbb{R}^d)$ can be written as

$$\begin{aligned}(x,y) &= (\bar{x}, \bar{y}) + \textstyle\sum_{i=1}^k \mu_i r^i \\ &= (\bar{x}, \bar{y}) + \textstyle\sum_{i=1}^k (\mu_i - \lfloor \mu_i \rfloor) r^i + \sum_{i=1}^k \lfloor \mu_i \rfloor r^i\end{aligned}$$

where $(\bar{x}, \bar{y}) \in K$ and $\mu_1, \ldots, \mu_t \geq 0$. Observe that

$$(\bar{x}, \bar{y}) + \sum_{i=1}^k (\mu_i - \lfloor \mu_i \rfloor) r^i \in \mathbb{Z}^n \times \mathbb{R}^d.$$

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Assume $P_i = \{x \in \mathbb{R}^n : A^i x \leq b^i\}$. Then

$$X = \left\{ x \in \mathbb{R}^n : \begin{array}{rcl} x & = & x^1 + \ldots + x^k + \mu_1 r^i + \ldots \mu_t r^t \\ & & A^i x^i \leq \delta_i b^i \\ & & \delta_1 + \ldots + \delta_k = 1 \\ & & \delta \in \mathbb{Z}_+^k, \mu \in \mathbb{Z}_+^t \end{array} \right\}$$

An alternate algebraic characterization?

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The family of Chvátal functions is the smallest family $\mathcal{F}$ of functions from $\mathbb{R}^n$ to $\mathbb{R}$ that contains all affine linear functions and is closed under

(i) finite nonnegative combinations, i.e., $f, g \in \mathcal{F}, \lambda, \gamma \geq 0 \Rightarrow \lambda f + \gamma g \in \mathcal{F}$, and

(ii) the operation of taking floors, i.e., $f \in \mathcal{F} \Rightarrow \lfloor f \rfloor \in \mathcal{F}$

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THEOREM (Basu-Ryan-Martin-Wang IPCO 2017): A set $X \subseteq \mathbb{R}^n$ is rationally MILP-representable if and only if

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$$\left\{ x \in \mathbb{R} : \begin{array}{l} \lfloor \frac{x-1}{2} \rfloor + \lfloor \frac{-x+2}{2} \rfloor \geq 0 \\ 1 \leq x \leq 4 \end{array} \right\}$$

Representability of Mixed-Integer Ellipsoidal Regions

DEFINITION: Any set of the form

$$E = \{x \in \mathbb{R}^d : (x - c)^T M(x - c) \leq \gamma\}$$

where $M \succeq 0$, $c \in \mathbb{R}^d$, and $\gamma > 0$, is called an ellipsoidal region. If $M \succ 0$ then E is an ellipsoid.

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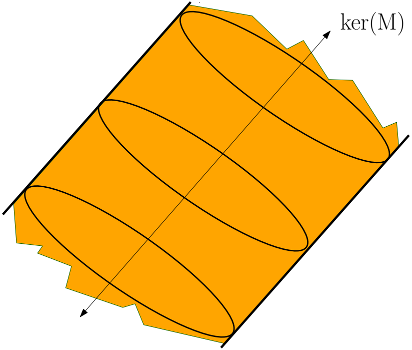
FACT: Any ellipsoidal region E is the Minkowski sum of an ellipsoid and $\mathrm{rec}(E) = \ker(M)$.



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We will call such sets (rationally) Ellipsoidal Mixed-Integer (EMI) representable.

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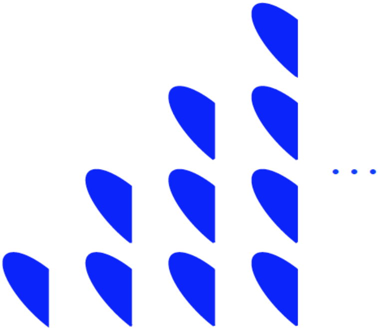

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Three issues:

- What is the recession cone of a set of the form $E \cap Q$?
- Is $K + \Pi$ of the form $E \cap P$?
- Is the projection of a set of the form $E \cap P$ again of the form $E' \cap P'$?

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Three issues:

- What is the recession cone of a set of the form $E \cap Q$?
  Answer: $\mathrm{rec}(E \cap Q) = \mathrm{rec}(E) \cap \mathrm{rec}(Q)$ as long as $E \cap Q \neq \emptyset$.
- Is $K + \Pi$ of the form $E \cap P$?
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Three issues:

- What is the recession cone of a set of the form $E \cap Q$?
- Is $K + \Pi$ of the form $E \cap P$? Answer: Yes, if one chooses K carefully. See Del Pia-Poskin paper for details.
- Is the projection of a set of the form $E \cap P$ again of the form $E' \cap P'$?

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$$C \cap (\mathbb{Z}^n \times \mathbb{R}^d) = \Big((K + \Pi) \cap (\mathbb{Z}^n \times \mathbb{R}^d)\Big) + \mathrm{int.cone}\{r^1, \ldots, r^t\},$$

where $\Pi = \{\sum_{i=1}^k \lambda_i r^i : 0 \leq \lambda_i \leq 1\}$.

Three issues:

- What is the recession cone of a set of the form $E \cap Q$?
- Is $K + \Pi$ of the form $E \cap P$?
- Is the projection of a set of the form $E \cap P$ again of the form $E' \cap P'$? I don't know! Main technical difficulty. Del Pia and Poskin show that it is of the form $\bigcup_{i=1}^k (E_i \cap P_i)$.

Representability of Mixed-Integer Ellipsoidal Regions

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THEOREM (Del Pia-Poskin IPCO 2016, MP 2018): A set $X \subseteq \mathbb{R}^n$ is rationally EMI-representable if and only if

$$X = \left(\bigcup_{i=1}^{k} (E_i \cap P_i) \right) + \operatorname{int.cone}\{r^1, \dots, r^t\}$$

where $E_1, \dots, E_k$ are rational ellipsoidal regions, $P_1, \dots, P_k$ are rational polytopes, and $r^1, \dots, r^t$ are integral vectors.

Representability of General Convex Mixed-Integer Sets

$\mathcal{S}$ is the family of sets given by the mixed-integer points in a general closed, convex set.

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Representability of Bilevel Optimization Problems

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$\mathcal{S}$ is the family of sets given by sets of the form

$$\left\{ (x,y) \in \mathbb{R}^m \times \mathbb{R}^d : \begin{array}{c} Ax + By \leq b \\ y \in \arg\max \left\{ c^T y : \begin{array}{l} Dy \leq d - Cx, \\ y \in \mathbb{Z}^{d_1} \times \mathbb{R}^{d_2} \end{array} \right\} \\ x_i \in \mathbb{Z}, \quad i \in I \subseteq \{1, \ldots, m\} \end{array} \right\}$$

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We say that a set $X \subseteq \mathbb{R}^n$ is mixed-integer bilevel representable if there exists $S \in \mathcal{S}$ and a linear transformation T such that $X = T(S)$.

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Rationality will play a role again.

Without integrality constraints first

$\tilde{\mathcal{S}}$ is the family of sets given by sets of the form

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Define the value function

$$V(x) \quad := \quad \max \left\{ c^T y : \begin{array}{l} Dy \leq d - Cx, \\ y \in \mathbb{R}^d \end{array} \right\}$$

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THEOREM (Basu-Ryan-Sankaranarayanan 2018): A set $X \subseteq \mathbb{R}^n$ is continuous bilevel representable if and only if X is a finite union of polyhedra.

Without integrality constraints first

Continuous Bilevel Optimization

$$\left\{ (x,y) \in \mathbb{R}^m \times \mathbb{R}^d : \begin{array}{c} Ax + By \leq b \\ y \in \arg\max \left\{ c^T y : \begin{array}{c} Dy \leq d - Cx, \\ y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \end{array} \right\} \end{array} \right\}$$

Linear Complementarity Problem

$$\left\{ x \in \mathbb{R}^n : \begin{array}{c} Ax \leq b \\ 0 \leq x \perp Mx + q \geq 0 \end{array} \right\}$$

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THEOREM (Basu-Ryan-Sankaranarayanan 2018): Let $X \subseteq \mathbb{R}^n$. Then the following are equivalent:

(i) X is continuous bilevel representable.

(ii) X is linear complementarity representable.

(iii) X is a finite union of polyhedra.

Add the integrality constraints

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Observation 1: If the integrality is added only in the upper level, then we get a union of MILP-representable sets.

$$\left\{ (x,y) \in \mathbb{R}^m \times \mathbb{R}^d : \begin{array}{l} Ax + By \leq b \\ y \in \arg\max \left\{ c^T y : \begin{array}{l} Dy \leq d - Cx, \\ y \in \mathbb{R}^d \end{array} \right\} \\ x_i \in \mathbb{Z}, \quad i \in I \subseteq \{1, \ldots, m\} \end{array} \right\}$$

COROLLARY (Basu-Ryan-Sankaranarayanan 2018): A set $X \subseteq \mathbb{R}^n$ is upper level integer bilevel representable if and only if X is a finite union of MILP-representable sets.

Add the integrality constraints

Observation 1: If the integrality is added only in the upper level, then we get a union of MILP-representable sets.

Observation 2: If the integrality is added in the lower level, then we may get a set that is not topologically closed even under rational data.

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Observation 2: If the integrality is added in the lower level, then we may get a set that is not topologically closed even under rational data.

EXAMPLE (Ryan-Koeppe-Queyranne JOTA 2010):

$$\left\{ (x,y) \in \mathbb{R} \times \mathbb{R} : \begin{array}{c} 0 \leq x \leq 1 \\ y \in \arg\max \left\{ y : \begin{array}{c} y \leq x, \\ 0 \leq y \leq 1 \\ y \in \mathbb{Z} \end{array} \right\} \end{array} \right\}$$

Add the integrality constraints

Mixed-Integer Bilevel Optimization

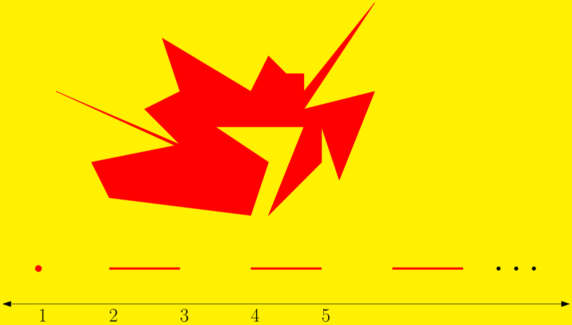
$$\left\{ (x,y) \in \mathbb{R}^m \times \mathbb{R}^d : \begin{array}{c} Ax + By \leq b \\ y \in \arg\max \left\{ c^T y : \begin{array}{l} Dy \leq d - Cx, \\ y \in \mathbb{Z}^{d_1} \times \mathbb{R}^{d_2} \end{array} \right\} \\ x_i \in \mathbb{Z}, \quad i \in I \subseteq \{1, \ldots, m\} \end{array} \right\}$$

THEOREM (Basu-Ryan-Sankaranarayanan 2018): Let $X \subseteq \mathbb{R}^n$. Then $X = \mathbf{cl}(S)$ for some $S \subseteq \mathbb{R}^n$ that is rational mixed-integer bilevel representable if and only if X is a finite union of rationally MILP-representable sets.

Add the integrality constraints

Mixe

CAUTION: A finite union of MILP-representable sets is not necessarily MILP-representable.



THE ... Then

$X =$ ... evel

repr

MIL

Proof of Main Theorem

THEOREM (Basu-Ryan-Sankaranarayanan 2018): Let $X \subseteq \mathbb{R}^n$. Then $X = \mathbf{cl}(S)$ for some $S \subseteq \mathbb{R}^n$ that is rationally mixed-integer bilevel representable if and only if X is a finite union of rationally MILP-representable sets.

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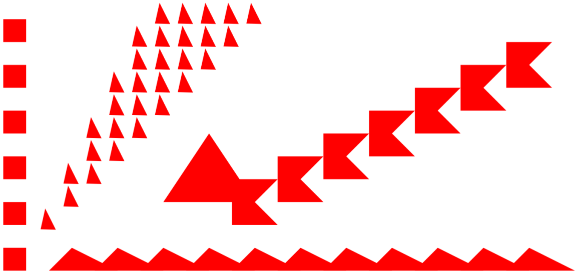

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PROOF:

$$\left\{ (x,y) \in \mathbb{R}^m \times \mathbb{R}^d : \begin{array}{c} Ax + By \leq b \\ y \in \arg\max \left\{ c^T y : \begin{array}{c} Dy \leq d - Cx, \\ y \in \mathbb{Z}^{d_1} \times \mathbb{R}^{d_2} \end{array} \right\} \\ x_i \in \mathbb{Z}, \quad i \in I \subseteq \{1, \ldots, m\} \end{array} \right\}$$

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PROOF:

$$\left\{ (x,y) \in \mathbb{R}^m \times \mathbb{R}^d : \begin{array}{l} Ax + By \leq b \\ c^T y \geq J(x) \\ x_i \in \mathbb{Z}, \quad i \in I \subseteq \{1, \ldots, m\} \end{array} \right\}$$

where $J(x)$ is the value function of a rational mixed-integer linear program.

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THEOREM (Blair/Jeroslow 1977, 1979, 1995): The value function of a rational MILP is of the form

$$J(x) = \max_{i \in I} \left\{ w_i^T (x - E_i \lfloor E_i^{-1} x \rfloor) + \min_{j \in J} \psi_j (E_i \lfloor E_i^{-1} x \rfloor) \right\}$$

where I, J are finite index sets, E_i, $i \in I$ are invertible matrices, and ψ_j, $j \in J$ are Chvátal functions. Such functions are called Jeroslow functions.

Proof of Main Theorem

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Bottomline: Need to analyze sub/super level sets of Chvátal functions.

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PROPOSITION (Basu-Ryan-Sankaranarayanan 2018): Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a rational Chvátal function. Then the closures of $\{x \in \mathbb{R}^n : \psi(x) \geq 0\}$, $\{x \in \mathbb{R}^n : \psi(x) \leq 0\}$, and $\{x \in \mathbb{R}^n : \psi(x) = 0\}$ are all finite unions of MILP-representable sets.

Sub/super level sets of Chvátal functions

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Recall algebraic characterization of MILP-representable sets: $\{x \in \mathbb{R}^n : \psi(x) \geq 0\}$ is a rational MILP-representable set.

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Recall algebraic characterization of MILP-representable sets: $\{x \in \mathbb{R}^n : \psi(x) \geq 0\}$ is a rational MILP-representable set.

Suffices to show that $\{x \in \mathbb{R}^n : \psi(x) \leq 0\}$ is a finite union of rationally MILP-representable sets (up to closures).

Sub/super level sets of Chvátal functions

PROPOSITION (Basu-Ryan-Sankaranarayanan 2018): Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a rational Chvátal function. Then the closures of $\{x \in \mathbb{R}^n : \psi(x) \geq 0\}$, $\{x \in \mathbb{R}^n : \psi(x) \leq 0\}$, and $\{x \in \mathbb{R}^n : \psi(x) = 0\}$ are all finite unions of MILP-representable sets.

Recall algebraic characterization of MILP-representable sets: $\{x \in \mathbb{R}^n : \psi(x) \geq 0\}$ is a rational MILP-representable set.

Suffices to show that $\{x \in \mathbb{R}^n : \psi(x) \leq 0\}$ is a finite union of rationally MILP-representable sets (up to closures).

Boils down to checking the following

LEMMA (Basu-Ryan-Sankaranarayanan 2018): Let X be a rational MILP-representable set. Then the complement of X is a finite union of rational MILP-representable sets (up to closures).

Complement of MILP-representable set

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Need to analyze

$$\left(\bigcup_{i=1}^{k} P_i + M\right)^c$$

Complement of MILP-representable set

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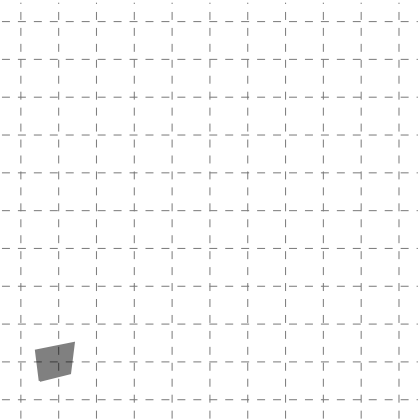
Need to analyze

$$\left(\bigcup_{i=1}^{k} P_i + M\right)^c = \left(\bigcup_{i=1}^{k} (P_i + M)\right)^c = \bigcap_{i=1}^{k} (P_i + M)^c$$

Since intersection of MILP-representable sets are MILP-representable sets, it suffices to show that given any polytope P and a finitely generated integral monoid M, the set $(P + M)^c$ is a finite union of rationally MILP-representable sets (up to closures).

$(P + M)^c$ is MILP-representable

First consider the case when M is generated by a linearly independent set of vectors.



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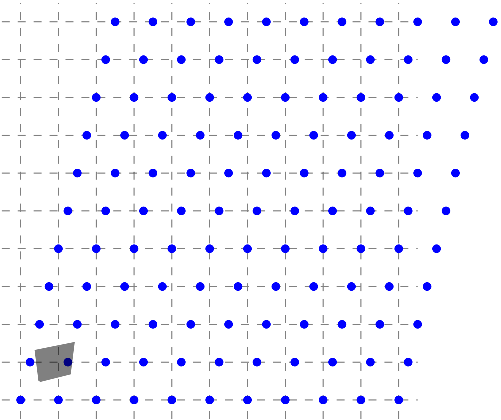

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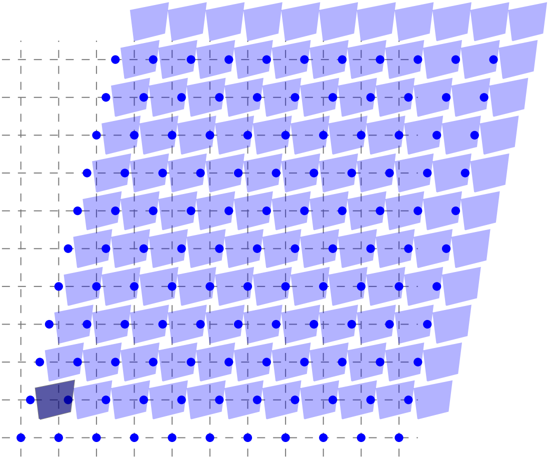

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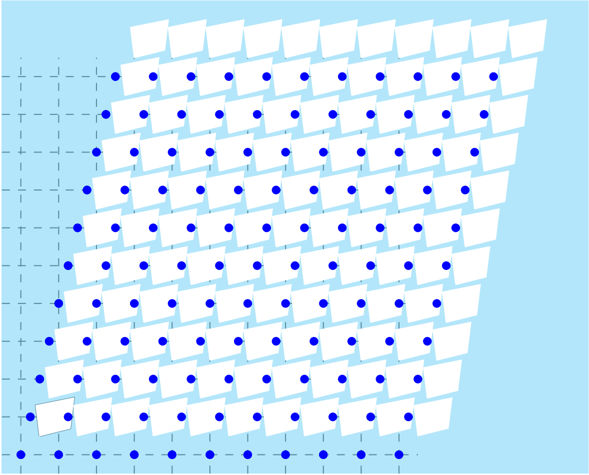

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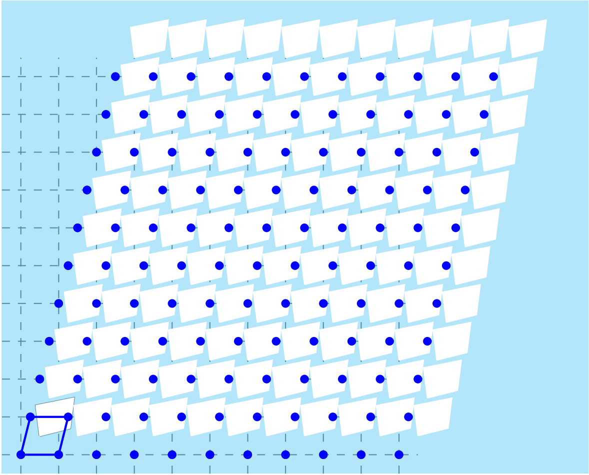

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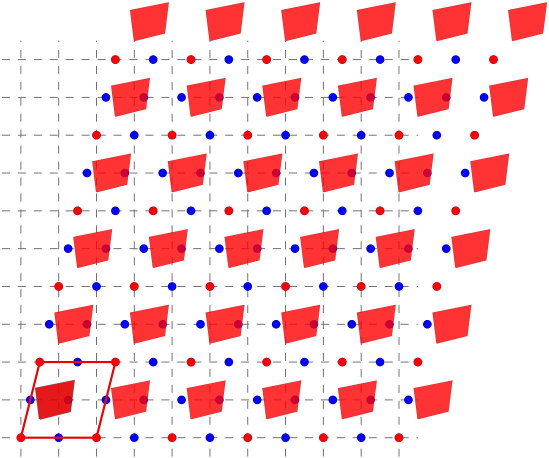

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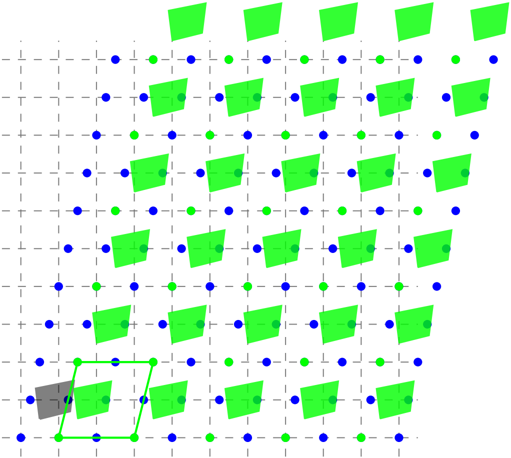

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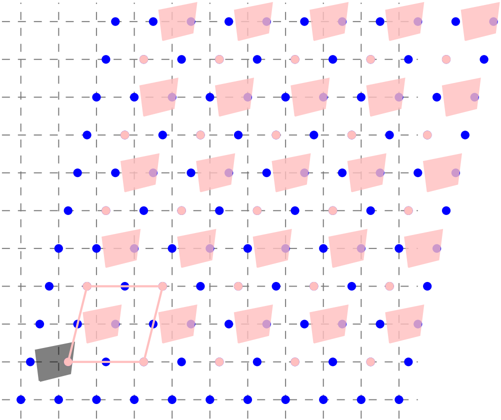

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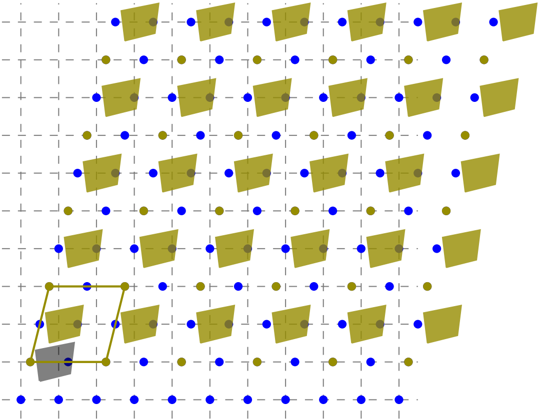

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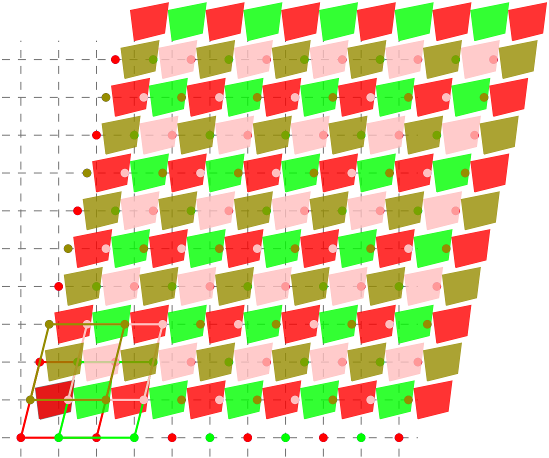


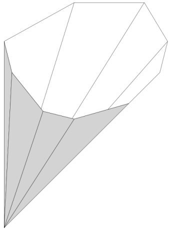
$(P + M)^c$ is MILP-representable

What if the finitely generated monoid M is not generated by linearly independent vectors?

$(P + M)^c$ is MILP-representable

What if the finitely generated monoid M is not generated by linearly independent vectors?

- Consider $C = \text{cone}(M)$. Write $C = \bigcup_{i=1}^{k} C_i$ where C_i are simplicial. Extreme rays of C_i are in M. Define $M_i = C_i \cap M$. Note that $M = \bigcup_{i=1}^{k} M_i$.



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What if the finitely generated monoid M is not generated by linearly independent vectors?

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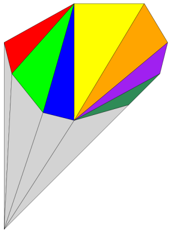

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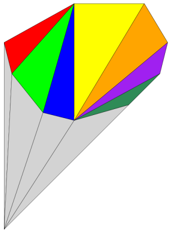
- By results of Jeroslow 1978, each M_i can be written as a finite union of monoids whose generators are extreme rays of C_i.



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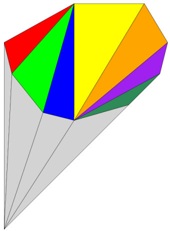
What if the finitely generated monoid M is not generated by linearly independent vectors?

- Consider $C = \text{cone}(M)$. Write $C = \bigcup_{i=1}^{k} C_i$ where C_i are simplicial. Extreme rays of C_i are in M. Define $M_i = C_i \cap M$. Note that $M = \bigcup_{i=1}^{k} M_i$.
- By results of Jeroslow 1978, each M_i can be written as a finite union of monoids whose generators are extreme rays of C_i.
- But since C_i are constructed to be simplicial, extreme rays of C_i are linearly independent. So each M_i is a finite union of monoids that are linearly independent.



Open Questions

- Sizes of bilevel formulations: Is there a MILP-representable subset of $\mathbb{R}^n$ that needs exponential (in n) sized MILP formulations, but has a polynomial size mixed-integer bilevel formulation? Can be asked about the hierarchy of n-level mixed-integer formulations.

- Representability of mixed-integer points in intersections of convex quadratic constraints.

THANK YOU !

Questions/Comments ?